ЗАКАРПАТСЬКИЙ УГОРСЬКИЙ ІНСТИТУТ ІМЕНІ ФЕРЕНЦА РАКОЦІ II
II. RÁKÓCZI FERENC KÁRPÁTALJAI MAGYAR FŐISKOLA

Кафедра математики та інформатики
Matematika és Informatika Tanszék

«ADDITIONAL TOPICS IN CONTEMPORARY MATHEMATICS»
(МЕТОДИЧНІ ВКАЗІВКИ ДЛЯ ПРАКТИЧНИХ РОБІТ)

(для студентів 2-го курсу спеціальності 014 Середня освіта (Математика))

ADDITIONAL TOPICS IN CONTEMPORARY MATHEMATICS
(Módszertani utmutató gyakorlati foglalkozásokhoz)

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(ступінь вищої освіти /a felsőoktatás szintje)

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Посібник з додаткових розділів сучасної математики призначений для студентів II курсу (ступеня магістра) Закарпатського угорського інституту імені Ференца Ракоці спеціальності 014 Середня освіта (математика) денної та заочної форми навчання з метою організації практичних занять з курсу "Додаткові розділи сучасної математики".

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Ez a jegyzet elsősorban matematika szakos hallgatók számára készült, de hasznos lehet mindazok számára, akik bármely más szakon tanulnak matematikát.

Az oktatási folyamatban történő felhasználását jóváhagyta
a II. Rákóczi Ferenc Kárpátaljai Magyar Főiskola
Matematika és Informatika Tanszéke
(2025. június 23, 10. számú jegyzőkönyv).

Megjelentetésre javasolta a II. Rákóczi Ferenc Kárpátaljai Magyar Főiskola
Minőségbiztosítási Tanácsa
(2025. augustzus 26, 7. számú jegyzőkönyv).

Elektronikus formában (PDF fájlformátumban) történő kiadásra javasolta
a II. Rákóczi Ferenc Kárpátaljai Magyar Főiskola Tudományos Tanácsa
(2025. augustzus 28, 8. számú jegyzőkönyv).

Kiadásra előkészítette a II. Rákóczi Ferenc Kárpátaljai Magyar Főiskola
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A tartalomért kizárólag a jegyzet szerkesztője felel.

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Contents

1. Group Theory

Part I. Multiple-Choice Questions

(Choose the most accurate option. Some may require detailed reasoning.)

1. Which of the following statements is **always** true for any group?
   a) Every element has a unique inverse.
   b) All groups are commutative.
   c) Every group has only one subgroup.
   d) Every finite group is cyclic.
2. Which description best fits the idea of a **normal subgroup**?
   a) A subgroup that contains all inverses of the group.
   b) A subgroup invariant under conjugation by any element of the group.
   c) A subgroup generated by a single element.
   d) A subgroup that is always abelian.
3. Which of the following captures the main consequence of **Lagrange's Theorem**?
   a) The order of every element must divide the order of the group.
   b) Every subgroup is normal.
   c) Every infinite group has an infinite cyclic subgroup.
   d) Groups cannot contain subgroups larger than half of their size.

Part II. Matching Definitions

(Match the concept with the correct definition.)

1. Homomorphism
2. Coset
3. Automorphism
4. Simple group
5. Group action

A. A structure-preserving map between two groups.
B. A subgroup multiplied by a fixed element of the group.
C. A group that cannot be broken down into smaller normal subgroups.
D. A way in which group elements can systematically transform another set.
E. An isomorphism from a group to itself.

Part III. Fill in the Gaps (Word Bank)

*(Choose from the words: **identity, cyclic, conjugation, permutation, subgroup, symmetry, orbit**).*

1. A _______ is a subset of a group that is itself a group.
2. The unique element that leaves all other elements unchanged under the operation is called the _______.
3. A group generated by one element is known as a _______ group.
4. When a group acts on a set, the set can be partitioned into disjoint _______.
5. The study of _______ in objects such as polygons and crystals is one of the key applications of group theory.
6. A _______ is a rearrangement of the elements of a set.

7. The process of transforming an element by surrounding it with another element and its inverse is called _______.

Part IV. Comparative Tasks

1. Compare **abelian** and **non-abelian** groups, focusing on structure, examples, and mathematical significance.
2. Compare **cosets** and **group orbits**: what unites them conceptually, and in which ways are they fundamentally different?
3. Compare **finite simple groups** with **infinite simple groups**: why are the first completely classified, while the second remain largely mysterious?

Part V. Analytical / Proof-Oriented Tasks

1. Explain why the intersection of two subgroups must itself be a subgroup, even without calculations.
2. Argue whether the union of two subgroups can form a subgroup. Provide reasoning and counterexamples in abstract terms.
3. Without using formulas, describe why every group of prime order must be cyclic.

Part VI. Advanced Applications

1. Discuss how group actions help solve combinatorial counting problems (for example, symmetry counting).
2. Analyze the role of group theory in **cryptography**. Why is the abstract structure of groups particularly useful in designing secure systems?
3. Examine how **symmetry groups** appear in physics, especially in particle theory or crystallography. Provide detailed reasoning.

Part VII. Open Discussion

1. Do you agree with the statement: *"Group theory is the foundation of modern abstract mathematics."*? Defend or refute with examples.
2. Can group theory explain every phenomenon of symmetry in nature, or are there limits to its applicability?

2. Ring Theory

Part I. Multiple-Choice Questions

(Choose the best option. In some cases, more than one answer may be correct.)

1. Which of the following statements is **always true** for every ring?
   a) Addition is commutative.
   b) Multiplication is commutative.
   c) Every non-zero element has an inverse.
   d) The additive identity exists.
2. Which of the following distinguishes a **commutative ring** from a general ring?
   a) The existence of an additive identity.
   b) The distributive property.
   c) The commutativity of multiplication.
   d) The existence of an absorbing element.
3. Which property characterizes an **integral domain**?
   a) It has no zero divisors.
   b) Every element has an additive inverse.
   c) Every element is invertible under multiplication.
   d) Its multiplication is non-associative.
4. Which of the following statements is correct about **ideals**?
   a) Every ideal is closed under multiplication with elements of the ring.
   b) Every ideal must contain the multiplicative identity.
   c) The intersection of two ideals is not an ideal.
   d) Every ring has only one ideal.

Part II. Matching Definitions

(Match the concept with its definition.)

1. Ideal
2. Zero divisor
3. Unit
4. Commutative ring
5. Field

A. A ring in which every non-zero element has a multiplicative inverse.
B. An element that has a multiplicative inverse inside the ring.
C. A ring in which multiplication is commutative.
D. An element (non-zero) that annihilates another non-zero element when multiplied.
E. A special subset of a ring that absorbs multiplication from the whole ring and is closed under addition.

Part III. Fill in the Gaps (Word Bank)

*(Choose from: **ideal, quotient, commutative, distributive, unit, zero divisor, polynomial**).*

1. A ring in which multiplication satisfies the property a(b+c)=ab+aca(b+c) = ab + aca(b+c)=ab+ac is called **_______**.

2. A non-zero element that annihilates another non-zero element under multiplication is a _______.
3. A _______ is an element that has a multiplicative inverse.
4. The set of cosets formed by dividing a ring by an ideal is known as a _______ ring.
5. The construction of _______ rings allows the study of algebraic equations within ring theory.
6. A subset of a ring that absorbs multiplication and is closed under addition is called an _______.
7. If the multiplication of a ring is also _______ , then the ring has a more specialized structure.

Part IV. Comparative Tasks

1. Compare **rings** and **fields**: what structural properties distinguish them, and why are these distinctions fundamental?
2. Compare **ideals** with **subgroups** in group theory: what is similar in their role, and what differs essentially?
3. Compare **integral domains** with **rings containing zero divisors**: what impact does this difference have on algebraic problem-solving?

Part V. Analytical / Proof-Oriented Tasks

1. Argue why every field is necessarily an integral domain, but not every integral domain is a field.
2. Explain why the intersection of two ideals is an ideal, but the union of two ideals need not be.
3. Provide reasoning why polynomial rings over fields play a central role in algebraic constructions.

Part VI. Advanced Applications

1. Discuss the role of ring theory in modern cryptography, especially in public-key systems.
2. Explain how ring theory provides the foundation for algebraic geometry through polynomial rings.
3. Analyze why the study of quotient rings is essential in understanding factorization and modular arithmetic.

Part VII. Open Discussion

1. Evaluate the statement: *"Rings generalize numbers while simultaneously unifying algebraic structures across mathematics."* Do you agree? Support your answer with examples.
2. Do you think that studying ideals in rings is analogous to studying normal subgroups in groups? Justify your reasoning.

3. Field Theory

Part I. Multiple-Choice Questions

(Choose the best option. More than one answer can be correct.)

1. Which of the following properties distinguishes a field from a ring?
   a) Every non-zero element has a multiplicative inverse.
   b) Multiplication is always commutative.
   c) Addition is not associative.
   d) Zero divisors exist.
2. Which of the following sets is **not** a field?
   a) Rational numbers with standard operations.
   b) Real numbers with standard operations.
   c) Integers with standard operations.
   d) Complex numbers with standard operations.
3. Which of the following is true about finite fields?
   a) They exist only for prime numbers of elements.
   b) They exist for any prime power of elements.
   c) Every finite field is isomorphic to some field of polynomials modulo an irreducible polynomial.
   d) There are infinitely many finite fields of the same size.

Part II. Matching Definitions

(Match the concept with the correct definition.)

1. Prime field
2. Characteristic of a field
3. Algebraic extension
4. Transcendental element
5. Splitting field

A. A field generated by adjoining all roots of a given polynomial.
B. A field with no proper subfields, usually the rationals or finite fields of prime order.
C. The smallest positive integer such that adding the identity element repeatedly yields zero.
D. An element that satisfies no polynomial equation with coefficients from the base field.
E. A larger field containing a smaller one, such that every element of the larger is a root of a polynomial over the smaller.

Part III. Fill in the Gaps (Word Bank)

*(Choose from: **inverse, extension, characteristic, algebraic, finite, prime, polynomial**).*

1. A field has the property that every non-zero element possesses a multiplicative _______.
2. The smallest number of times the unity must be added to itself to yield zero is called the _______ of the field.
3. A field with no proper subfields is referred to as a _______ field.
4. The process of creating a larger field from a smaller one is known as a field _______.
5. A number is called _______ over a field if it is the root of some non-zero _______ with coefficients from that field.

6. Every **_______** field has order equal to a power of a prime.

Part IV. Comparative Tasks

1. Compare **fields** with **rings**: why does the existence of multiplicative inverses for all non-zero elements fundamentally change their structure?
2. Compare **algebraic extensions** with **transcendental extensions**: what conceptual boundary separates them?
3. Compare **finite fields** with **infinite fields**: explain differences in construction, properties, and applications.

Part V. Analytical / Proof-Oriented Tasks

1. Explain why the field of rational numbers is the smallest field containing the integers.
2. Without formulas, argue why a finite field must contain a number of elements equal to a power of a prime.
3. Provide a logical explanation of why every polynomial has a splitting field, and why this concept is central in field theory.

Part VI. Advanced Applications

1. Analyze the role of field theory in the development of Galois theory and its implications for solving polynomial equations.
2. Discuss how finite fields are used in coding theory and error correction.
3. Explore how field extensions form the algebraic foundation for modern cryptography.

Part VII. Open Discussion

1. Do you agree that fields represent the most "balanced" algebraic structures, combining the properties of addition and multiplication in their purest form? Defend your position.
2. Evaluate the statement: *"Without field theory, modern algebra and number theory could not exist as we know them."* Support your reasoning with examples.

4. Module Theory

Part I. Multiple-Choice Questions

(Choose the most accurate option. Some may require detailed reasoning.)

1. Which of the following best describes a **module**?
   a) A generalization of a vector space where scalars come from an arbitrary ring.
   b) A group with a multiplication operation that is always commutative.
   c) A set closed only under scalar multiplication.
   d) A subgroup of a field with extra structure.
2. Which of the following statements is true about modules?
   a) Every module over a field is a vector space.
   b) Every module is necessarily free.
   c) All modules must have a finite basis.
   d) Modules can only be defined over commutative rings.
3. Which of the following is **not** necessarily a submodule?
   a) The intersection of two submodules.
   b) The sum of two submodules.
   c) The union of two submodules.
   d) The trivial submodule containing only the zero element.
4. Which condition distinguishes a **free module** from a general module?
   a) Existence of a linearly independent generating set.
   b) Closure under scalar multiplication.
   c) Closure under addition.
   d) Existence of zero divisors in the ring.

Part II. Matching Definitions

(Match the concept with its correct definition.)

1. Submodule
2. Free module
3. Cyclic module
4. Homomorphism of modules
5. Quotient module

A. A module generated by a single element.
B. A module where each element can be uniquely expressed as a linear combination of basis elements.
C. A subset of a module closed under addition and scalar multiplication.
D. The structure obtained by dividing a module by one of its submodules.
E. A function between modules preserving addition and scalar multiplication.

Part III. Fill in the Gaps (Word Bank)

*(Choose from: **basis, quotient, generator, submodule, homomorphism, free, cyclic**).*

1. A _______ of a module is an element whose multiples generate the whole module.
2. A _______ is a module that has a set of elements forming a basis similar to vector spaces.
3. A _______ module is generated by a single element.

4. The natural mapping from a module to its _______ module is always a homomorphism.
5. A _______ is a subset of a module that itself carries the structure of a module.
6. A function preserving scalar multiplication and addition between modules is a _______.
7. A collection of elements forming a minimal generating set for a free module is called a _______.

Part IV. Comparative Tasks

1. Compare **modules** and **vector spaces**: why is the generalization to modules both powerful and more complex?
2. Compare **submodules** and **ideals**: how do they play similar structural roles in modules and rings?
3. Compare **cyclic modules** and **cyclic groups**: where is the analogy precise, and where does it fail?

Part V. Analytical / Proof-Oriented Tasks

1. Without using formulas, argue why every module over a field must be a vector space.
2. Explain why the intersection of two submodules is again a submodule, but the union may not be.
3. Discuss the significance of free modules: why are they considered the building blocks of module theory?

Part VI. Advanced Applications

1. Analyze how module theory extends linear algebra beyond vector spaces, especially in cases where scalars come from rings with zero divisors.
2. Discuss the importance of modules in representation theory and homological algebra.
3. Explore the connection between modules and systems of linear equations over rings.

Part VII. Open Discussion

1. Evaluate the statement: *"Modules are the true generalization of vector spaces, and therefore they represent the core of modern algebra."* Do you agree?
2. In your view, what makes module theory more challenging than group, ring, or field theory? Provide a reasoned argument.

5. Galois Theory

Part I. Multiple-Choice Questions

(Choose the most accurate option. Some may require detailed reasoning.)

1. Which of the following best describes the **fundamental theorem of Galois theory**?
   a) A correspondence between normal subgroups of a group and its quotient groups.
   b) A bijection between intermediate fields of a field extension and subgroups of its Galois group.
   c) A proof that every polynomial has roots in the complex numbers.
   d) A classification of finite groups according to their order.
2. Which of the following statements is always true about Galois groups?
   a) They are necessarily abelian.
   b) They consist of all automorphisms of a field extension that fix the base field.
   c) They always act trivially on the extension field.
   d) They exist only for finite extensions.
3. Which extensions are typically studied in Galois theory?
   a) Arbitrary extensions of infinite degree.
   b) Algebraic extensions that are both normal and separable.
   c) Transcendental extensions only.
   d) Finite extensions without restriction.
4. Which of the following problems motivated the birth of Galois theory?
   a) The classification of all finite groups.
   b) The impossibility of solving general polynomial equations of degree five and higher by radicals.
   c) The construction of transcendental numbers.
   d) The foundations of set theory.

Part II. Matching Definitions

(Match the concept with its correct definition.)

1. Normal extension
2. Separable extension
3. Galois group
4. Splitting field
5. Radical extension

A. An extension in which a polynomial decomposes completely into linear factors.
B. A group of automorphisms of an extension that leave the base field fixed.
C. An extension obtained by adjoining successive roots of radicals.
D. An extension where every irreducible polynomial with a root in it splits completely.
E. An extension in which each element is the root of a polynomial with distinct roots.

Part III. Fill in the Gaps (Word Bank)

*(Choose from: **automorphism, normal, separable, correspondence, radical, splitting, symmetry**).*

1. Galois theory studies field extensions through their _______ groups.
2. A Galois extension must be both _______ and _______.

3. The fundamental theorem establishes a **_______** between intermediate fields and subgroups.
4. A **_______** extension is one where a polynomial completely decomposes.
5. Galois theory reveals the deep connection between algebraic equations and **_______**.
6. A **_______** extension is built by adjoining successive roots of radicals.

Part IV. Comparative Tasks

1. Compare **normal extensions** and **separable extensions**: how do they differ, and why does Galois theory require both conditions?
2. Compare the **splitting field** of a polynomial with its **radical extension**: when do they coincide, and when do they diverge?
3. Compare the role of **automorphisms** in Galois groups with the role of **symmetries** in geometry.

Part V. Analytical / Proof-Oriented Tasks

1. Without formulas, explain why the fundamental theorem of Galois theory creates a bridge between algebra and group theory.
2. Argue why not every polynomial equation is solvable by radicals, relating your reasoning to the structure of the Galois group.
3. Discuss why the symmetry structure of the roots of a polynomial dictates the solvability of the equation.

Part VI. Advanced Applications

1. Analyze the role of Galois theory in proving the insolvability of the general quintic equation.
2. Discuss how Galois theory connects to modern cryptography and coding theory.
3. Explore how Galois groups provide insight into classical geometric problems (e.g., constructibility with ruler and compass).

Part VII. Open Discussion

1. Evaluate the statement: *"Galois theory transformed algebra from a computational tool into a structural science."* Do you agree? Why?
2. In your view, is the connection between symmetries of polynomials and group theory the most profound idea in modern mathematics? Support your argument.

6. Quaternion Theory

Part I. Multiple-Choice Questions

(Choose the most accurate option. Some may require detailed reasoning.)

1. Which statement best characterizes the algebra of quaternions?
   a) A commutative field extending the complex numbers.
   b) A four-dimensional non-commutative division algebra over the reals.
   c) A finite abelian group with four generators.
   d) A purely geometric construction unrelated to algebra.
2. Which of the following properties distinguishes quaternions from real and complex numbers?
   a) Lack of additive inverses.
   b) Non-associativity of multiplication.
   c) Non-commutativity of multiplication.
   d) Absence of multiplicative identity.
3. Which area of mathematics or physics most directly benefits from quaternion representation?
   a) Number theory.
   b) Cryptography.
   c) Spatial rotations and three-dimensional geometry.
   d) Real analysis.
4. Which of the following is true about the norm of a quaternion?
   a) It is always non-negative and multiplicative.
   b) It depends only on the imaginary part.
   c) It is not preserved under multiplication.
   d) It cannot be defined without transcendental functions.

Part II. Matching Definitions

(Match the concept with its correct definition.)

1. Quaternion algebra
2. Unit quaternion
3. Conjugate quaternion
4. Division algebra
5. Non-commutativity

A. An algebra in which every nonzero element has a multiplicative inverse.
B. A quaternion of norm equal to one, often used to represent spatial rotations.
C. An operation reversing the sign of the imaginary components.
D. A structure extending complex numbers with four-dimensional basis over the reals.
E. A property where the order of multiplication matters.

Part III. Fill in the Gaps (Word Bank)

*(Choose from: **rotation, conjugate, non-commutative, norm, unit, division**).*

1. A quaternion algebra is a four-dimensional **_______** algebra over the real numbers.
2. The **_______** of a quaternion is always non-negative and multiplicative.

3. A _______ quaternion is used to encode spatial transformations in three dimensions.
4. Taking the _______ of a quaternion changes the signs of its imaginary components.
5. Quaternion multiplication is _______; the order of factors matters.
6. Quaternions are powerful in representing _______ in physics and computer graphics.

Part IV. Comparative Tasks

1. Compare **complex numbers** and **quaternions**: in what sense are quaternions a natural extension, and in what sense are they fundamentally different?
2. Compare **rotation matrices** and **unit quaternions**: which advantages do quaternions bring in practice?
3. Compare the role of **quaternion conjugation** with that of **complex conjugation**: how are they analogous, and how do they diverge?

Part V. Analytical / Proof-Oriented Tasks

1. Explain why quaternion multiplication is associative but not commutative.
2. Justify why every nonzero quaternion has a multiplicative inverse, despite non-commutativity.
3. Discuss why the norm of a quaternion is preserved under multiplication and why this makes quaternions useful in geometry.

Part VI. Advanced Applications

1. Analyze the importance of quaternions in representing three-dimensional rotations in robotics and aerospace engineering.
2. Discuss how quaternion interpolation (slerp) is applied in computer graphics and animation.
3. Explore the historical role of quaternions in the development of vector analysis.

Part VII. Open Discussion

1. Evaluate the statement: *"Quaternions are the most natural language for three-dimensional geometry, even more so than matrices."* Do you agree? Argue your position.
2. In your opinion, why did quaternions not replace vectors in mainstream mathematics, despite their elegance?

7. Group Ring Theory

Part I. Multiple-Choice Questions

(Choose the most accurate option. Some require deep reasoning.)

1. What is the defining feature of a group ring?
   a) It is a direct sum of subgroups.
   b) It combines the structure of a ring with the formal linear combinations of group elements.
   c) It is a ring generated only by abelian groups.
   d) It coincides with the polynomial ring in every case.
2. Which of the following best describes elements of a group ring?
   a) Arbitrary mappings from a group to a ring.
   b) Formal finite sums of group elements with coefficients from a ring.
   c) Infinite series indexed by group elements.
   d) Pairs consisting of a subgroup and a coefficient ring.
3. Which property holds in every group ring?
   a) Multiplication is commutative whenever the coefficient ring is commutative.
   b) Multiplication is commutative regardless of the group.
   c) The structure is always a field if the group is finite.
   d) The group ring always has zero divisors.
4. Which problem area has been historically linked to group rings?
   a) The classification of finite simple groups.
   b) The isomorphism problem for group rings.
   c) The foundations of set theory.
   d) The unsolvability of quintic equations.

Part II. Matching Definitions

(Match each concept with its correct definition.)

1. Group ring
2. Group algebra
3. Augmentation map
4. Integral group ring
5. Zero divisor

A. A homomorphism that maps each group element to one, extending linearly.
B. A structure built from a group and a commutative ring by forming linear combinations.
C. A group ring with integers as coefficients.
D. An algebra over a field formed from a group.
E. A nonzero element of a ring whose product with another nonzero element equals zero.

Part III. Fill in the Gaps (Word Bank)

*(Choose from: **augmentation, coefficients, linear, zero-divisors, integral, multiplication, homomorphism**).*

1. Elements of a group ring are formal sums with _______ from a ring.
2. The operation of _______ extends the group structure to the ring structure.
3. The _______ map assigns value one to each group element.

4. An _______ group ring arises when the coefficient ring is the ring of integers.
5. Group rings may contain _______ even when the coefficient ring is an integral domain.
6. The augmentation map is a ring _______.

Part IV. Comparative Tasks

1. Compare **group rings** and **polynomial rings**: what structural similarities exist, and where do they diverge?
2. Compare the **augmentation map** in group rings with the **evaluation homomorphism** in polynomial rings.
3. Compare group algebras over fields with group rings over the integers: which additional difficulties appear in the integral case?

Part V. Analytical / Proof-Oriented Tasks

1. Explain why a group ring can be seen as a bridge between abstract algebra (group theory) and linear algebra (vector spaces/modules).
2. Justify why the non-commutativity of the group may force the non-commutativity of the group ring, even if the coefficient ring is commutative.
3. Argue why the presence of zero-divisors in group rings makes them more difficult to study than simple polynomial rings.

Part VI. Advanced Applications

1. Discuss the role of group rings in representation theory: how do modules over group rings connect to group representations?
2. Analyze the isomorphism problem for group rings: why is it significant, and what are the challenges?
3. Explore the applications of group rings in coding theory and cryptography.

Part VII. Open Discussion

1. Evaluate the statement: *"Group rings unify discrete symmetry (groups) with algebraic structure (rings), creating one of the most versatile tools in modern algebra."* Do you agree? Support your answer.
2. Why do you think the study of group rings remains challenging, despite their apparently simple definition?

8. Group Representation Theory

Part I. Multiple-Choice Questions

(Choose the most accurate option. Some may require deep reasoning.)

1. What is the main idea of group representation theory?
   a) To classify groups according to their order.
   b) To study groups by representing their elements as linear transformations of vector spaces.
   c) To find all normal subgroups of a group.
   d) To compute the determinant of a group.
2. Which of the following statements is true about a representation of a group?
   a) Every representation is injective.
   b) Every group element corresponds to an invertible linear transformation.
   c) Representations exist only for finite groups.
   d) The image of a representation is always abelian.
3. Which of the following concepts is central in the study of representations?
   a) Character theory.
   b) Ideal theory.
   c) Zero divisors.
   d) Quotient topology.
4. Which of the following statements is correct about irreducible representations?
   a) They have no proper, non-trivial invariant subspaces under the group action.
   b) They exist only for abelian groups.
   c) They can be decomposed further into smaller representations.
   d) They are always one-dimensional.

Part II. Matching Definitions

(Match the concept with its correct definition.)

1. Representation
2. Irreducible representation
3. Character of a representation
4. Invariant subspace
5. Direct sum of representations

A. A vector space that remains unchanged under the action of every group element.
B. A homomorphism from a group to the group of invertible linear transformations of a vector space.
C. A representation that cannot be decomposed into smaller non-trivial representations.
D. The trace function of the linear transformations corresponding to group elements.
E. A combination of two or more representations acting independently on separate subspaces.

Part III. Fill in the Gaps (Word Bank)

*(Choose from: **linear, homomorphism, trace, irreducible, invariant, direct sum, character**).*

1. A group representation is a **_______** mapping from a group to linear transformations of a vector space.
2. An **_______** representation has no proper invariant subspaces.

3. The _______ of each linear transformation in a representation defines its character.
4. A subspace that remains unchanged under the action of all group elements is called _______.
5. Multiple representations can be combined into a _______ of representations.
6. The character is a function that assigns to each group element a number derived from the _______ of its action.

Part IV. Comparative Tasks

1. Compare **group representations** with **modules over group rings**: how are these approaches connected?
2. Compare **irreducible representations** with **simple modules**: what is the precise analogy?
3. Compare **characters** with **traces of linear maps**: why is this connection fundamental in representation theory?

Part V. Analytical / Proof-Oriented Tasks

1. Explain why studying representations over complex vector spaces is often easier than over arbitrary fields.
2. Argue why every finite-dimensional representation of a finite group over the complex numbers can be decomposed into a direct sum of irreducible representations.
3. Discuss why character tables provide complete information about the representations of finite groups.

Part VI. Advanced Applications

1. Analyze the role of group representation theory in quantum mechanics, particularly in studying symmetries of physical systems.
2. Explore how representation theory is applied in crystallography and molecular symmetry.
3. Discuss the application of group characters in counting and combinatorial problems.

Part VII. Open Discussion

1. Evaluate the statement: *"Group representation theory transforms abstract groups into concrete objects, making them more tangible and analyzable."* Do you agree? Explain your reasoning.
2. In your view, why is representation theory considered a bridge between algebra and other areas of mathematics and physics?

9. Matrix Representations of Groups

Part I. Multiple-Choice Questions

(Choose the most accurate option. Some may require reasoning.)

1. What is the main idea of a matrix representation of a group?
   a) Assigning a unique number to each group element.
   b) Representing group elements as invertible matrices acting on a vector space.
   c) Decomposing groups into subgroups.
   d) Representing group elements as points in a geometric space.
2. Which of the following statements is true about matrix representations?
   a) Every representation is commutative.
   b) Matrix representations make abstract group operations concrete.
   c) Only finite groups can be represented by matrices.
   d) All matrices in a representation must be diagonalizable.
3. Which concept is central in the study of matrix representations of finite groups?
   a) Eigenvalues of individual matrices.
   b) Character theory.
   c) Topological invariants.
   d) Ring homomorphisms.
4. Which statement about irreducible matrix representations is correct?
   a) They have proper invariant subspaces.
   b) They cannot be decomposed into smaller invariant subspaces.
   c) They are always one-dimensional.
   d) They exist only for abelian groups.

Part II. Matching Definitions

(Match the concept with its correct definition.)

1. Matrix representation
2. Irreducible representation
3. Character of a representation
4. Invariant subspace
5. Equivalent representations

A. Two representations are considered equivalent if they differ only by a change of basis.
B. A subspace of a vector space that remains invariant under all matrices of the representation.
C. A linear mapping from a group to invertible matrices over a field.
D. A representation that cannot be decomposed into smaller invariant subspaces.
E. A function assigning the trace of the matrix corresponding to each group element.

Part III. Fill in the Gaps (Word Bank)

*(Choose from: **trace, basis, invertible, invariant, equivalent, irreducible, matrices**).*

1. A matrix representation assigns _______ to each group element.
2. A subspace that remains unchanged under all matrices is called _______.
3. An _______ representation has no proper invariant subspaces.
4. Two representations that differ only by a change of _______ are called equivalent.

5. The character of a matrix representation is defined as the **_______** of the matrices corresponding to group elements.
6. All matrices in a representation are required to be **_______** (have inverses).

Part IV. Comparative Tasks

1. Compare **matrix representations** and **abstract group representations**: why are matrices often preferred for computations?
2. Compare **irreducible representations** and **direct sum decompositions**: how does the concept of irreducibility simplify analysis?
3. Compare **characters** in matrix representations with **trace functions**: why is the trace an invariant under equivalence?

Part V. Analytical / Proof-Oriented Tasks

1. Explain why every finite-dimensional representation of a finite group over the complex numbers can be decomposed into a direct sum of irreducible representations.
2. Discuss why equivalent matrix representations carry the same character.
3. Argue why matrix representations allow one to apply linear algebra techniques to study group properties.

Part VI. Advanced Applications

1. Analyze the role of matrix representations in quantum mechanics, especially in describing symmetries of physical systems.
2. Discuss how characters of matrix representations help classify molecules in chemistry.
3. Explore the application of matrix representations in crystallography and computer graphics.

Part VII. Open Discussion

1. Evaluate the statement: *"Matrix representations provide the most tangible link between abstract groups and concrete computations."* Do you agree? Explain.
2. Why is the study of matrix representations fundamental for connecting group theory with linear algebra and applications in physics?

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10. Modern Approaches in the Methodology of Teaching Mathematics and Informatics

Part I. Case Analysis

1. A mathematics teacher notices that students can solve problems mechanically but cannot explain their reasoning. Discuss which modern teaching approaches could improve critical thinking and understanding.
2. In an informatics class, students learn programming syntax but struggle to design solutions for real problems. Propose a teaching plan using project-based or problem-based methods to address this.
3. Compare two scenarios: one class uses traditional lectures only, the other uses flipped classroom with interactive exercises. Identify potential differences in student engagement and learning outcomes.

Part II. Short Analytical Tasks

1. Explain how **computational thinking** can be integrated into a high school mathematics lesson without relying on programming.
2. Identify three advantages and three challenges of using technology (simulations, virtual labs) in mathematics and informatics lessons.
3. Describe how **formative assessment** can be implemented during a single practical session to guide student learning.

Part III. Comparative and Classification Exercises

1. Compare **traditional lecture-based approach**, **project-based learning**, and **flipped classroom** in terms of:
    - student autonomy
    - teacher’s role
    - collaboration opportunities
2. Classify the following activities as **student-centered** or **teacher-centered**:
    - Students discuss solutions in small groups.
    - Teacher explains the method while students take notes.
    - Students work on a mini-project simulating a real-life problem.
    - Teacher gives a quiz with predetermined answers.

Part IV. Fill in the Gaps / Word Bank

*(Choose from: **interactive learning, collaborative projects, problem-solving, adaptive learning, digital tools, flipped classroom, critical thinking**).*

1. Modern methodology emphasizes _______ to engage students actively in the lesson.
2. Lessons using real-life problems encourage _______ skills in both mathematics and informatics.
3. Teaching that adapts pace and difficulty to student performance is called _______.
4. Lessons using videos or pre-recorded materials at home and active tasks in class implement a _______.
5. Group work and shared tasks are typical examples of _______.
6. Software, simulations, and educational apps are considered _______ in modern lessons.

7. Analysis, evaluation, and reasoning are part of developing _______ skills.

Part V. Group Discussion / Debate

1. Debate: *"Traditional methods are still more effective than modern approaches for learning mathematics."* Prepare arguments for and against.
2. Discuss: *"Technology in informatics classes improves learning only if combined with active problem-solving."* Provide examples.
3. Reflect on which modern teaching approach you would choose for a mixed-ability class and justify your choice.

Part VI. Practical Simulation / Microteaching

1. Design a 15-minute micro-lesson using **project-based learning** to teach a mathematical concept to peers. Include objectives, student tasks, and assessment strategy.
2. Create a short **interactive activity** using a digital tool (simulation, spreadsheet, or coding exercise) for a practical informatics class. Explain how it fosters engagement and understanding.
3. Propose a **group problem-solving session** in mathematics: define the problem, assign roles, and describe how students report and discuss solutions.

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