ЗАКАРПАТСЬКИЙ УГОРСЬКИЙ ІНСТИТУТ ІМЕНІ ФЕРЕНЦА РАКОЦІ II
II. RÁKÓCZI FERENC KÁRPÁTALJAI MAGYAR FŐISKOLA

Кафедра математики та інформатики
Matematika és Informatika Tanszék

«ADDITIONAL TOPICS IN CONTEMPORARY MATHEMATICS»
(МЕТОДИЧНІ ВКАЗІВКИ ДЛЯ САМОСТІЙНОЇ РОБОТИ)

(для студентів 2-го курсу спеціальності 014 Середня освіта (Математика))

ADDITIONAL TOPICS IN CONTEMPORARY MATHEMATICS
(Módszertani utmutató önállói munkához)

Другий (магістерський) / Mesterképzés (MA)
(ступінь вищої освіти /a felsőoktatás szintje)

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(галузь знань / képzési ág)

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(освітня програма / képzési program)

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Посібник з додаткових розділів сучасної математики призначений для студентів ІІ курсу (ступеня магістра) Закарпатського угорського інституту імені Ференца Ракоці спеціальності 014 Середня освіта (математика) денної та заочної форми навчання з метою організації самостійної роботи з курсу "Додаткові розділи сучасної математики".

Матеріал призначений для використання як навчально-методичний посібник з дисципліни "Додаткові розділи сучасної математики".

Затверджено до використання у навчальному процесі
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Ez a jegyzet elsősorban matematika szakos hallgatók számára készült, de hasznos lehet mindazok számára, akik bármely más szakon tanulnak matematikát.

Az oktatási folyamatban történő felhasználását jóváhagyta
a II. Rákóczi Ferenc Kárpátaljai Magyar Főiskola
Matematika és Informatika Tanszéke
(2025. június 23, 10. számú jegyzőkönyv).

Megjelentetésre javasolta a II. Rákóczi Ferenc Kárpátaljai Magyar Főiskola
Minőségbiztosítási Tanácsa
(2025. augustzus 26, 7. számú jegyzőkönyv).

Elektronikus formában (PDF fájlformátumban) történő kiadásra javasolta
a II. Rákóczi Ferenc Kárpátaljai Magyar Főiskola Tudományos Tanácsa
(2025. augustzus 28, 8. számú jegyzőkönyv).

Kiadásra előkészítette a II. Rákóczi Ferenc Kárpátaljai Magyar Főiskola
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A tartalomért kizárólag a jegyzet szerkesztője felel.

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Contents

1. Group Theory

Part I. Multiple-Choice Questions

(Choose the most accurate option. Some may require reasoning.)

1. Which of the following best defines a group?
   a) A set with a single associative operation and an identity element.
   b) A set with a binary operation that is closed, associative, has an identity, and every element has an inverse.
   c) A set with an operation that is always commutative.
   d) A set of numbers closed under addition.
2. Which statement about subgroups is true?
   a) Every subset of a group is a subgroup.
   b) A subgroup must contain the identity element and be closed under the group operation and inverses.
   c) Subgroups are only defined for abelian groups.
   d) Subgroups must have the same order as the original group.
3. What distinguishes a normal subgroup from a general subgroup?
   a) It is abelian.
   b) It is invariant under conjugation by any element of the group.
   c) It contains only the identity element.
   d) It must be finite.
4. Which of the following statements is true about quotient groups?
   a) They are always isomorphic to the original group.
   b) They are formed by partitioning a group using a normal subgroup.
   c) They are defined only for finite groups.
   d) They contain no identity element.

Part II. Matching Definitions

(Match the concept with its correct definition.)

1. Group
2. Subgroup
3. Normal subgroup
4. Coset
5. Quotient group

A. A set of elements of the form gH where g belongs to G and H is a subgroup.
B. A group where a specific subgroup is invariant under conjugation.
C. A set with a binary operation satisfying closure, associativity, identity, and invertibility.
D. A subgroup of a group.
E. The set of cosets of a normal subgroup forms a group under a well-defined operation.

Part III. Fill in the Gaps (Word Bank)

*(Choose from: **identity, inverse, associative, normal, coset, subgroup, quotient**).*

1. Every element of a group has a unique _______ under the group operation.
2. A binary operation is _______ if the order of operations does not affect the outcome.
3. A _______ subgroup is invariant under conjugation by any element of the group.
4. The set of all elements combined with a subgroup to form gH is called a _______.
5. A _______ is a subset of a group that itself forms a group under the same operation.
6. Partitioning a group by a normal subgroup produces a _______ group.
7. The element that leaves other elements unchanged under the group operation is called the _______.

Part IV. Comparative Tasks

1. Compare **subgroups** and **normal subgroups**: why are normal subgroups crucial for forming quotient groups?
2. Compare **left cosets** and **right cosets**: when do they coincide?
3. Compare **quotient groups** and the original group: what structural information is preserved?

Part V. Analytical / Proof-Oriented Tasks

1. Without formulas, explain why the identity element in a group is unique.
2. Argue why inverses in a group are unique.
3. Discuss why every subgroup of an abelian group is automatically normal.
4. Explain why the set of cosets forms a well-defined group only when the subgroup is normal.

Part VI. Advanced Applications

1. Discuss the role of group theory in solving puzzles like the Rubik's cube.
2. Explore how group theory underlies symmetry in chemistry and physics.
3. Analyze how group theory informs the study of permutation groups in combinatorics.

Part VII. Open Discussion

1. Evaluate the statement: *"Group theory provides the language of symmetry in mathematics and physics."* Do you agree? Explain.
2. In your view, why is the concept of a normal subgroup fundamental to modern algebra?

2. Ring Theory

Part I. Multiple-Choice Questions

(Choose the most accurate option. Some require deep reasoning.)

1. Which statement correctly defines a ring?
   a) A set with addition only, satisfying closure and associativity.
   b) A set with two operations, addition and multiplication, where addition forms an abelian group, multiplication is associative, and multiplication distributes over addition.
   c) A set where multiplication is commutative but addition is not defined.
   d) A set with an identity element for addition only.
2. Which of the following is true about commutative rings?
   a) All rings are automatically commutative.
   b) The multiplication operation satisfies commutativity for all elements.
   c) Only rings with a finite number of elements can be commutative.
   d) Commutativity only applies to addition.
3. What is an ideal in ring theory?
   a) A subset of a ring closed under subtraction and multiplication by any element of the ring.
   b) Any subgroup of the additive group of the ring.
   c) A subset closed under multiplication only.
   d) A set containing only the multiplicative identity.
4. Which statement about quotient rings is correct?
   a) They are defined by partitioning a ring by any subset.
   b) They are formed by partitioning a ring using an ideal, inheriting ring structure.
   c) They exist only for commutative rings.
   d) They cannot have a multiplicative identity.

Part II. Matching Definitions

(Match each concept with its correct definition.)

1. Ring
2. Commutative ring
3. Ideal
4. Quotient ring
5. Multiplicative identity

A. A subset of a ring such that multiplying any element of the ring by an element of the subset yields another element in the subset.
B. A ring where multiplication is commutative.
C. An element that leaves other elements unchanged under multiplication.
D. The set of cosets formed from a ring and an ideal, inheriting ring operations.
E. A set with two operations, addition and multiplication, satisfying group and distributive properties.

Part III. Fill in the Gaps (Word Bank)

*(Choose from: **addition, multiplication, ideal, commutative, distributive, quotient, identity**).*

1. In a ring, _______ forms an abelian group.
2. The operation of _______ is associative and may or may not be commutative.
3. An _______ is a special subset that allows the construction of a quotient ring.
4. A ring is called _______ if multiplication is commutative for all elements.
5. The law connecting addition and multiplication in a ring is called the _______ property.
6. Partitioning a ring by an ideal produces a _______ ring.
7. The element that leaves others unchanged under multiplication is the _______ element.

Part IV. Comparative Tasks

1. Compare **commutative rings** and **non-commutative rings**: which properties are preserved and which differ?
2. Compare **ideals** and **subgroups**: in what sense are ideals generalizations of subgroups?
3. Compare **quotient rings** and **quotient groups**: what structural parallels exist?

Part V. Analytical / Proof-Oriented Tasks

1. Explain why the additive identity in a ring is unique.
2. Argue why the distributive property is essential in defining a ring.
3. Discuss why every ring with unity allows for the definition of principal ideals.
4. Explain why the quotient by an ideal always forms a ring.

Part VI. Advanced Applications

1. Analyze the role of ring theory in number theory, such as in modular arithmetic.
2. Discuss applications of ring theory in coding theory and cryptography.
3. Explore how rings are used in algebraic geometry to study structures defined by polynomials.

Part VII. Open Discussion

1. Evaluate the statement: *"Rings provide a unifying framework for studying both addition and multiplication structures in mathematics."* Do you agree? Explain.
2. Why are ideals considered central in modern algebra, especially in relation to quotient structures?

3. Field Theory

Part I. Multiple-Choice Questions

(Choose the most accurate option. Some require reasoning.)

1. Which of the following best defines a field?
   a) A set with only an addition operation that is associative.
   b) A set with addition and multiplication, where both operations form groups, multiplication is commutative, and multiplication distributes over addition (excluding zero for multiplication).
   c) A set with only a multiplication operation that is commutative.
   d) A set where only addition is invertible.
2. Which statement about finite fields is true?
   a) All finite fields have the same number of elements.
   b) The order (number of elements) of a finite field is always a prime power.
   c) Finite fields exist only for prime numbers.
   d) Finite fields cannot be used in cryptography.
3. What is the characteristic of a field?
   a) The number of additive inverses in the field.
   b) The smallest positive number of times one must add the multiplicative identity to itself to get zero, or zero if no such number exists.
   c) The total number of elements in the field.
   d) The number of nonzero elements that are invertible.
4. Which of the following is true about field extensions?
   a) They always produce finite fields.
   b) They are larger fields containing a smaller field as a subfield.
   c) They only exist for real numbers.
   d) They cannot be used to study polynomials.

Part II. Matching Definitions

(Match the concept with its correct definition.)

1. Field
2. Subfield
3. Characteristic
4. Finite field
5. Field extension

A. A field containing a smaller field as a subset with inherited operations.
B. A field with a finite number of elements.
C. A number indicating the additive property of the identity element with respect to repeated addition.
D. A set with two operations (addition and multiplication) satisfying field axioms.
E. A larger field containing a smaller field to allow new elements and operations.

Part III. Fill in the Gaps (Word Bank)

*(Choose from: **subfield, extension, characteristic, multiplicative, additive, prime, finite**).*

1. A _______ is a smaller field contained within a larger field.
2. A _______ adds new elements to a field while preserving field operations.
3. The _______ of a field is the smallest number of times the multiplicative identity must be added to itself to produce zero.
4. All elements of a field except zero have a _______ inverse.
5. The additive identity in any field forms a group under _______ operation.
6. A field with a prime or prime-power number of elements is called a _______ field.
7. A field with a finite number of elements is called a _______ field.

Part IV. Comparative Tasks

1. Compare **fields** and **rings**: which additional properties distinguish a field from a general ring?
2. Compare **subfields** and **field extensions**: what is the conceptual difference?
3. Compare **finite fields** and **infinite fields**: how does the structure and application differ?

Part V. Analytical / Proof-Oriented Tasks

1. Explain why every nonzero element of a field has a unique multiplicative inverse.
2. Argue why the characteristic of a field must be either zero or a prime number.
3. Discuss why field extensions are crucial for understanding solutions of polynomials.
4. Explain how the concept of subfields helps classify finite and infinite fields.

Part VI. Advanced Applications

1. Discuss the role of finite fields in coding theory and cryptography.
2. Explore how field extensions are used in solving classical geometric problems, such as constructions with ruler and compass.
3. Analyze the importance of field theory in algebraic number theory and polynomial factorization.

Part VII. Open Discussion

1. Evaluate the statement: *"Fields form the foundation of modern algebra, connecting numbers, geometry, and algebraic structures."* Do you agree? Explain.
2. Why is the study of field extensions fundamental for understanding solvability of equations?

4. Module Theory

Part I. Multiple-Choice Questions

(Choose the most accurate option. Some require reasoning.)

1. Which statement best defines a module over a ring?
   a) A module is a vector space where scalars come from a field.
   b) A module is an additive abelian group equipped with a scalar multiplication by elements of a ring, satisfying certain axioms.
   c) A module is any group with two operations.
   d) A module is a ring with additional vector operations.
2. Which of the following is true about submodules?
   a) Every subset of a module is a submodule.
   b) A submodule is a subset that is closed under addition and scalar multiplication by elements of the ring.
   c) Submodules exist only for modules over fields.
   d) Submodules are always isomorphic to the original module.
3. What distinguishes a free module from other modules?
   a) It has a basis such that every element is a unique linear combination of basis elements with coefficients from the ring.
   b) It is always finite-dimensional.
   c) It exists only over commutative rings.
   d) Its additive group is non-abelian.
4. Which statement about quotient modules is correct?
   a) They are formed by partitioning a module using any subset.
   b) They are formed by partitioning a module using a submodule, inheriting module structure.
   c) Quotient modules exist only for modules over fields.
   d) They cannot have additive identity.

Part II. Matching Definitions

(Match each concept with its correct definition.)

1. Module
2. Submodule
3. Free module
4. Quotient module
5. Homomorphism of modules

A. A structure-preserving map between two modules.
B. A module with a basis allowing unique representation of elements via ring coefficients.
C. A subset of a module that itself forms a module under the same operations.
D. A module formed by partitioning a module using a submodule.
E. An additive abelian group with scalar multiplication by a ring satisfying compatibility axioms.

Part III. Fill in the Gaps (Word Bank)

*(Choose from: **submodule, free, homomorphism, quotient, scalar, abelian, basis**).*

1. A _______ is a subset of a module that is itself a module under the same operations.
2. A module is called _______ if it has a set of generators forming a basis.
3. A _______ preserves addition and scalar multiplication when mapping one module to another.
4. Partitioning a module by a submodule produces a _______ module.
5. Modules generalize vector spaces where scalars come from a _______ instead of a field.
6. The additive group of a module is always _______.
7. Elements in a free module can be expressed uniquely as linear combinations of _______ elements.

Part IV. Comparative Tasks

1. Compare **modules** and **vector spaces**: what properties differ and what is generalized?
2. Compare **submodules** and **ideals**: what is the analogous structure in ring theory?
3. Compare **free modules** and **quotient modules**: how does the concept of generators vs. cosets affect their structure?

Part V. Analytical / Proof-Oriented Tasks

1. Explain why every submodule of a module over a field is automatically a vector space.
2. Argue why the additive identity in a module is unique.
3. Discuss why homomorphisms of modules are central to understanding module structure.
4. Explain how quotient modules generalize the idea of quotient groups and quotient rings.

Part VI. Advanced Applications

1. Discuss the role of module theory in linear algebra over rings instead of fields.
2. Explore applications of modules in representation theory of groups and algebras.
3. Analyze how module theory underlies concepts in algebraic topology and homological algebra.

Part VII. Open Discussion

1. Evaluate the statement: *"Modules generalize vector spaces and allow algebraic structures to interact with rings beyond fields."* Do you agree? Explain.
2. Why is the concept of free modules crucial for understanding generators and relations in algebra?

5. Galois Theory

Part I. Multiple-Choice Questions

(Choose the most accurate option. Some require reasoning.)

1. Which statement best describes Galois Theory?
   a) A study of symmetries of polynomial roots using group theory and field theory.
   b) A method of solving differential equations using linear algebra.
   c) A branch of topology analyzing continuous transformations.
   d) A theory focusing on sequences and series in number theory.
2. What is a Galois group?
   a) The set of all subfields of a given field.
   b) The group of field automorphisms of an extension field that fix the base field.
   c) A subgroup of a ring under multiplication.
   d) The additive group of a vector space over a field.
3. Which of the following is true about Galois extensions?
   a) They exist only for finite fields.
   b) They are field extensions for which the field automorphisms correspond exactly to intermediate fields.
   c) They always have trivial automorphism groups.
   d) They are defined only for polynomial equations of degree two.
4. What is the fundamental theorem of Galois Theory concerned with?
   a) Classifying all finite groups.
   b) Establishing a one-to-one correspondence between subgroups of the Galois group and intermediate fields of the extension.
   c) Determining the characteristic of a field.
   d) Solving linear equations in modules.

Part II. Matching Definitions

(Match each concept with its correct definition.)

1. Galois extension
2. Galois group
3. Intermediate field
4. Automorphism
5. Fundamental theorem of Galois Theory

A. A field lying between the base field and the extended field in a field extension.
B. The group of field automorphisms fixing the base field.
C. A field extension with properties linking its automorphisms and intermediate fields.
D. A bijective mapping of a field onto itself preserving operations.
E. A theorem establishing a correspondence between subgroups of the Galois group and intermediate fields.

Part III. Fill in the Gaps (Word Bank)

*(Choose from: **automorphism, extension, subgroup, intermediate, correspondence, polynomial, solvable**).*

1. A field **_______** adds new elements to a base field while maintaining field operations.
2. A **_______** is a bijective map of a field preserving addition and multiplication.
3. The group of all automorphisms of an extension field fixing the base field is called the Galois **_______**.
4. A field lying between the base field and the extension is an **_______** field.
5. The fundamental theorem of Galois Theory establishes a one-to-one **_______** between subgroups and intermediate fields.
6. Galois Theory analyzes the solvability of polynomial equations in terms of the structure of their Galois groups.
7. A **_______** of a Galois group corresponds to a subfield of the extension field.

Part IV. Comparative Tasks

1. Compare **Galois groups** and **general groups**: how does the restriction of fixing a base field influence the structure?
2. Compare **intermediate fields** and **subgroups of a Galois group**: why does the fundamental theorem create a correspondence?
3. Compare **solvable polynomials** and **non-solvable polynomials**: how does the structure of the Galois group determine solvability by radicals?

Part V. Analytical / Proof-Oriented Tasks

1. Explain why the Galois group of a trivial extension (field equals base field) is the trivial group.
2. Argue why every automorphism in the Galois group preserves the solutions of the defining polynomial.
3. Discuss why the correspondence between subgroups and intermediate fields is inclusion-reversing.
4. Explain how Galois Theory connects field theory and group theory to solve classical problems in algebra.

Part VI. Advanced Applications

1. Discuss how Galois Theory proves the impossibility of solving the general quintic by radicals.
2. Explore applications of Galois Theory in cryptography and coding theory.
3. Analyze how Galois Theory underlies modern algebraic number theory and algebraic geometry.

Part VII. Open Discussion

1. Evaluate the statement: *"Galois Theory provides the deepest connection between algebraic structures and the solvability of equations."* Do you agree? Explain.
2. Why is the concept of the Galois group considered central to understanding polynomial equations?

6. Quaternion Theory

Part I. Multiple-Choice Questions

(Choose the most accurate option. Some require reasoning.)

1. Which statement best defines quaternions?
   a) A set of numbers with only real components and commutative multiplication.
   b) A number system extending complex numbers with one real part and three imaginary parts, with non-commutative multiplication.
   c) A two-dimensional vector space over the reals.
   d) A field in which multiplication is always commutative.
2. What distinguishes quaternion multiplication from real or complex multiplication?
   a) It is associative and commutative.
   b) It is associative but generally non-commutative.
   c) It is commutative but not associative.
   d) It is neither associative nor distributive.
3. Which property do quaternions always satisfy?
   a) Existence of additive identity only.
   b) Existence of additive and multiplicative inverses (except for zero).
   c) Commutativity of multiplication for all elements.
   d) Being a field in the classical sense.
4. How are quaternions commonly applied?
   a) Modeling planar transformations only.
   b) Representing rotations in three-dimensional space.
   c) Solving polynomial equations of degree five.
   d) Generating only one-dimensional vector spaces.

Part II. Matching Definitions

(Match each concept with its correct definition.)

1. Quaternion
2. Conjugate
3. Norm of a quaternion
4. Unit quaternion
5. Non-commutative multiplication

A. A quaternion with norm equal to one, often representing a rotation.
B. The operation reversing the sign of the imaginary components.
C. A number system extending complex numbers with one real part and three imaginary parts.
D. The property where multiplication order affects the result.
E. A scalar value representing the "length" or magnitude of a quaternion.

Part III. Fill in the Gaps (Word Bank)

*(Choose from: **non-commutative, three, conjugate, norm, rotation, inverse, unit**).*

1. Quaternion multiplication is generally **_______**.
2. A quaternion consists of one real part and **_______** imaginary parts.
3. The **_______** of a quaternion is obtained by reversing the sign of its imaginary components.

4. The _______ of a quaternion measures its magnitude.
5. A quaternion representing a rotation in 3D space is called a _______ quaternion.
6. Every nonzero quaternion has a unique multiplicative _______.
7. Quaternions are widely used to describe rotations in _______ dimensions.

Part IV. Comparative Tasks

1. Compare **quaternions** and **complex numbers**: how does the addition of extra imaginary components affect algebraic properties?
2. Compare **unit quaternions** and **general quaternions**: what special role do unit quaternions play in applications?
3. Compare **commutative multiplication** in real/complex numbers and **non-commutative multiplication** in quaternions: how does it affect computations?

Part V. Analytical / Proof-Oriented Tasks

1. Explain why quaternion multiplication is non-commutative using the idea of imaginary components.
2. Argue why every nonzero quaternion has a multiplicative inverse.
3. Discuss why the conjugate of a quaternion is essential for computing its norm.
4. Explain how unit quaternions can be used to compose rotations without gimbal lock.

Part VI. Advanced Applications

1. Discuss the role of quaternions in computer graphics and animation.
2. Explore the use of quaternions in robotics and spacecraft orientation.
3. Analyze how quaternions relate to modern physics, particularly in representing rotations and spin.

Part VII. Open Discussion

1. Evaluate the statement: *"Quaternions provide a superior way to model 3D rotations compared to Euler angles."* Do you agree? Explain.
2. Why is non-commutativity a crucial feature in quaternion theory, especially for applications in three-dimensional rotation?

7. Group Ring Theory

Part I. Multiple-Choice Questions

(Choose the most accurate option. Some require reasoning.)

1. Which statement best describes a group ring?
   a) A set of groups with multiplication only.
   b) A structure formed by combining a group and a ring, where elements are formal sums of group elements with coefficients from the ring.
   c) A vector space over a field with addition only.
   d) A set with two operations that must always be commutative.
2. Which of the following is true about elements of a group ring?
   a) They are individual group elements without coefficients.
   b) They are sums of group elements multiplied by coefficients from a ring.
   c) They cannot be added.
   d) They always form a field.
3. Which property distinguishes group rings from ordinary rings?
   a) The underlying ring must be finite.
   b) The ring structure incorporates the multiplicative structure of a group.
   c) The group elements are ignored in multiplication.
   d) Group rings are always commutative.
4. What is the relevance of a group ring in algebra?
   a) It is used to classify vector spaces over finite fields.
   b) It serves as a tool to study representations of groups and module structures.
   c) It is primarily used to solve differential equations.
   d) It only applies to abelian groups.

Part II. Matching Definitions

(Match each concept with its correct definition.)

1. Group ring
2. Coefficient ring
3. Group element
4. Module over a group ring
5. Convolution multiplication

A. The ring providing coefficients for linear combinations of group elements.
B. A structure combining a ring and a group where elements are formal sums with multiplication respecting the group law.
C. A structure allowing the study of representations of the group via modules.
D. The operation combining sums of group elements and coefficients according to group and ring operations.
E. Individual elements belonging to the underlying group.

Part III. Fill in the Gaps (Word Bank)

*(Choose from: **coefficients, formal sum, module, convolution, abelian, non-commutative, representation**).*

1. An element of a group ring is a _______ of group elements with ring coefficients.
2. The set providing scalars for the formal sums is called the _______ ring.
3. A module over a group ring can be used to study group _______.
4. The multiplication in a group ring is defined using _______ multiplication of elements.
5. Group rings over _______ groups may have commutative multiplication, but in general they are non-commutative.
6. Group rings generalize both group theory and ring theory by combining group elements with _______.
7. Studying modules over a group ring allows understanding of linear _______ of the group.

Part IV. Comparative Tasks

1. Compare **group rings** and **ordinary rings**: how does adding the group structure influence properties?
2. Compare **modules over a group ring** and **vector spaces over a field**: what is generalized, and what new complexities appear?
3. Compare **commutative group rings** and **non-commutative group rings**: how does the group structure affect multiplication?

Part V. Analytical / Proof-Oriented Tasks

1. Explain why the additive identity in a group ring is unique.
2. Argue why every group element naturally embeds into the group ring as a basis element.
3. Discuss how convolution multiplication respects both the group law and the ring law.
4. Explain how studying modules over group rings allows us to classify group representations.

Part VI. Advanced Applications

1. Discuss the role of group rings in representation theory and character theory of finite groups.
2. Explore applications of group rings in algebraic topology and homological algebra.
3. Analyze how group rings are used in coding theory and cryptography.

Part VII. Open Discussion

1. Evaluate the statement: *"Group rings unify group theory and ring theory to provide a framework for studying linear representations."* Do you agree? Explain.
2. Why is non-commutativity in group rings significant for the structure of modules and representations?

8. Group Representation Theory

Part I. Multiple-Choice Questions

(Choose the most accurate option. Some require reasoning.)

1. What is a group representation?
   a) A mapping from a group to another group that preserves multiplication.
   b) A homomorphism from a group to the group of linear transformations of a vector space.
   c) A set of group elements with added scalar coefficients.
   d) A field extension associated with a group.
2. Which of the following is true about irreducible representations?
   a) They can always be decomposed into smaller nontrivial representations.
   b) They cannot be decomposed into smaller representations and serve as building blocks for all representations.
   c) They exist only for abelian groups.
   d) They are always one-dimensional.
3. What does the character of a representation describe?
   a) The sum of all group elements.
   b) A function assigning to each group element the trace of its representing matrix.
   c) The determinant of the representation matrices.
   d) The dimension of the underlying vector space only.
4. Which of the following statements is true about representations of finite groups over complex numbers?
   a) All representations are necessarily one-dimensional.
   b) Every representation can be decomposed into a direct sum of irreducible representations.
   c) Representations are always trivial.
   d) Representation theory does not apply to finite groups.

Part II. Matching Definitions

(Match each concept with its correct definition.)

1. Group representation
2. Irreducible representation
3. Character of a representation
4. Direct sum of representations
5. Homomorphism

A. A mapping from a group to linear transformations preserving the group operation.
B. A representation that cannot be decomposed into smaller nontrivial representations.
C. The function assigning the trace of the representing matrix to each group element.
D. A construction combining representations so that each element acts independently on each summand.
E. A structure-preserving map between groups.

Part III. Fill in the Gaps (Word Bank)

*(Choose from: **trace, homomorphism, irreducible, vector, decomposition, direct sum, representation**).*

1. A group representation is a _______ from a group to linear transformations of a vector space.
2. A representation that cannot be further broken into smaller invariant subspaces is called _______.
3. The _______ of a representation assigns a scalar to each group element corresponding to the trace of its matrix.
4. Representations of finite groups over the complex numbers can often be expressed as a _______ of irreducible representations.
5. The space on which a representation acts is called a _______ space.
6. The operation combining multiple representations so that each acts independently is called a _______ of representations.
7. Studying characters allows understanding of representation structure without explicit matrices, using the _______ function.

Part IV. Comparative Tasks

1. Compare **irreducible representations** and **general representations**: why are irreducibles considered the building blocks?
2. Compare **group representations** and **group homomorphisms**: how do vector spaces extend the concept of homomorphisms?
3. Compare **direct sum of representations** and **tensor product of representations**: how do these constructions differ in combining representations?

Part V. Analytical / Proof-Oriented Tasks

1. Explain why the character of a representation is constant on conjugacy classes of the group.
2. Argue why every finite-dimensional representation of a finite group over the complex numbers can be decomposed into irreducible components.
3. Discuss the significance of Schur's Lemma in classifying irreducible representations.
4. Explain how character tables summarize crucial information about group representations.

Part VI. Advanced Applications

1. Discuss the role of group representation theory in quantum mechanics and particle physics.
2. Explore applications of representations in crystallography and molecular symmetry.
3. Analyze the importance of character tables in computational group theory and chemistry.

Part VII. Open Discussion

1. Evaluate the statement: *"Group representations translate abstract groups into concrete linear actions, making their structure more tangible."* Do you agree? Explain.
2. Why is the study of irreducible representations central to understanding the full representation theory of a group?

9. Matrix Representations of Groups

Part I. Multiple-Choice Questions

(Choose the most accurate option. Some require reasoning.)

1. What is a matrix representation of a group?
   a) A mapping from a group to matrices over a field, preserving the group operation via matrix multiplication.
   b) A representation of matrices as abstract groups.
   c) A vector space with no group action.
   d) A group homomorphism into a ring of polynomials.
2. Which property is essential for a matrix representation?
   a) Matrix addition must correspond to the group operation.
   b) Matrix multiplication must reflect the group multiplication.
   c) Matrices must be diagonal.
   d) Matrices must commute for all group elements.
3. Which statement is true about irreducible matrix representations?
   a) They can always be decomposed into smaller invariant subspaces.
   b) They cannot be decomposed into smaller invariant subspaces.
   c) They exist only for abelian groups.
   d) They are always one-dimensional matrices.
4. What does the dimension of a matrix representation indicate?
   a) The number of group elements.
   b) The dimension of the vector space on which the group acts.
   c) The number of subgroups in the group.
   d) The determinant of all matrices.

Part II. Matching Definitions

(Match each concept with its correct definition.)

1. Matrix representation
2. Irreducible matrix representation
3. Character of a matrix representation
4. Direct sum of representations
5. Similarity of representations

A. A matrix representation that cannot be decomposed into smaller invariant subspaces.
B. A mapping from a group to invertible matrices preserving the group multiplication.
C. A function assigning the trace of each representing matrix to the corresponding group element.
D. Two representations are similar if one can be transformed into the other via a change of basis.
E. A construction combining representations so that each acts independently on a summand.

Part III. Fill in the Gaps (Word Bank)

*(Choose from: **trace, invertible, vector, direct sum, similarity, irreducible, homomorphism**).*

1. A matrix representation is a **_______** from a group to invertible matrices over a field.
2. A representation that cannot be further broken into invariant subspaces is called **_______**.

3. The _______ of a representation maps each group element to the trace of its representing matrix.
4. Representations can be combined using the _______ operation so that each acts independently.
5. Two representations are _______ if one can be converted into the other by a basis change.
6. Matrix representations act on a _______ space associated with the group.
7. Each matrix in a representation must be _______ to preserve group structure.

Part IV. Comparative Tasks

1. Compare **matrix representations** and **abstract group representations**: how does using matrices provide concrete computations?
2. Compare **irreducible matrix representations** and **reducible ones**: why are irreducibles considered fundamental?
3. Compare **similarity of representations** and **equivalence of representations**: how does change of basis affect the classification?

Part V. Analytical / Proof-Oriented Tasks

1. Explain why the character of a matrix representation is constant on conjugacy classes.
2. Argue why every finite-dimensional representation of a finite group over a field of characteristic zero can be decomposed into irreducible matrix representations.
3. Discuss the significance of Schur's Lemma in the context of matrix representations.
4. Explain how similarity transformations preserve the essential properties of matrix representations.

Part VI. Advanced Applications

1. Discuss the role of matrix representations in quantum mechanics for describing symmetries of particles.
2. Explore applications in crystallography and molecular symmetry analysis.
3. Analyze how matrix representations facilitate computations in computational group theory and chemistry.

Part VII. Open Discussion

1. Evaluate the statement: *"Matrix representations transform abstract group theory into concrete linear algebra, enabling computational applications."* Do you agree? Explain.
2. Why is the study of irreducible and similar matrix representations essential for classifying group actions?

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10. Modern Approaches in the Methodology of Teaching Mathematics and Informatics

Part I. Multiple-Choice Questions

(Choose the most accurate option.)

1. Which of the following best describes the modern approach to teaching mathematics?
   a) Focus exclusively on memorization of formulas.
   b) Emphasize problem-solving, critical thinking, and real-world applications.
   c) Teaching only through lectures without interaction.
   d) Avoid using technology in the classroom.
2. What is a key feature of modern informatics education?
   a) Only theoretical knowledge is required.
   b) Integration of programming, data analysis, and computational thinking.
   c) Focus solely on hardware installation skills.
   d) Exclusion of collaborative projects.
3. Which teaching method promotes active student participation and collaboration?
   a) Traditional lecture-based instruction
   b) Project-based learning
   c) Memorization exercises
   d) Teacher-centered lecturing
4. What role does technology play in modern mathematics and informatics education?
   a) It is optional and rarely useful.
   b) It facilitates interactive learning, simulations, and problem-solving.
   c) It replaces all teaching activities.
   d) It only serves as a grading tool.

Part II. Matching Definitions

(Match each concept with its correct definition.)

1. Problem-Based Learning (PBL)
2. Flipped Classroom
3. Computational Thinking
4. Interactive Learning
5. Formative Assessment

A. A method where students actively solve real-world problems as a primary learning tool.
B. A teaching strategy where students learn content at home and engage in interactive activities in class.
C. The process of thinking algorithmically and logically to solve problems using computers.
D. Learning that involves active participation, collaboration, and feedback.
E. Ongoing assessment used to monitor learning and guide instruction.

Part III. Fill in the Gaps (Word Bank)

*(Choose from: **collaboration, simulation, critical thinking, technology, project-based learning, adaptive learning, student-centered**).*

1. Modern teaching emphasizes _______ approaches rather than passive listening.
2. Use of educational software allows students to explore mathematical models through _______.
3. Teaching methods encourage _______ to analyze and evaluate solutions.
4. Incorporating computers and digital tools enhances learning through _______.
5. Students engage in hands-on tasks and real-world projects through _______.
6. _______ methods adjust the pace and content based on individual student needs.
7. Modern methodology prioritizes a _______ approach to teaching, focusing on active learning.

Part IV. Comparative Tasks

1. Compare **traditional lecture-based teaching** and **project-based learning**: which skills are better developed in each approach?
2. Compare **flipped classroom** and **interactive learning**: how does the role of the teacher and student change?
3. Compare **formative assessment** and **summative assessment**: how does each impact student learning?

Part V. Analytical / Proof-Oriented Tasks

1. Explain why integrating computational thinking is crucial for modern mathematics and informatics education.
2. Discuss the advantages and challenges of using technology in the classroom.
3. Analyze how active learning strategies influence student engagement and performance.
4. Argue the importance of adapting teaching methods to different learning styles and student needs.

Part VI. Advanced Applications

1. Explore the use of virtual labs and simulations in teaching complex mathematical concepts.
2. Discuss the role of collaborative online platforms in informatics education.
3. Analyze how project-based learning prepares students for real-world problem-solving in STEM fields.

Part VII. Open Discussion

1. Evaluate the statement: *"Modern approaches in teaching mathematics and informatics improve critical thinking and problem-solving skills more effectively than traditional methods."* Do you agree? Explain.
2. How can teachers balance technology use with traditional instructional methods to maximize learning outcomes?

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